Coefficients Of The Hilbert Polynomial Of A Determinental Ideal

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1 Abstract

What follows is a survey of Kulkarni's paper "Counting Of Paths and Coefficients Of Hilbert Polynomial Of A Determinental Ideal". This paper essentially expands on the previous work ("Enumerative Combinatorics Of Young Tableaux") by Abhyankar. In that work, Abhyankar found that the size of a special set of monomials is equal to the Hilbert polynomial evaluated at a certain point. In this paper, Kulkarni finds a correspondence between these coefficients and a special family of lattice paths.

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2 Introduction

Before launching into a theoretic description of the paper, I will attempt to lay down a road map for what will follow.

In essence, Abhyankar's work leaves us with the following pertinent facts (note: I will give the proper definitions later).

1. F(v) = $|mon((m,n),p,\tilde{a},v)|$

2. F(v) = $\sum_{e=0}^{\infty} H_e((m,n),p,\tilde{a},e)\ \binom{v-e+c}{c}$

3. Given a family of sets $\{X_v\}_{v=0}^{\infty}$ and a family of pairwise disjoint finite sets $\{Y_e\}_{e=0}^{\infty}$, if there exists a function $\phi : \cup X_v \to \cup Y_e$ so that for any $e, v \in \mathbb{N}$ and any $y \in Y_e$ the size of the preimage of y in X_v is dependent only on e and v (call this size $\lambda(v,e)$), and if $\{f(v)\}_{v=0}^{\infty}$ is a family of polynomials in $\mathbb{Q}[v]$ satisfying $f(v) = |X_v|$ and $f(v) = \sum_{e=0}^{\infty} h_e \lambda(v,e)$ (where the λ are a family of $\mathbb{Q}$-linearly independent polynomials in $\mathbb{Q}[v]$ if we let e go from 0 to ∞), then $|Y_e| = h_e\ \forall\ e \in \mathbb{N}$.

In our case, we would like to show that $H_e((m,n),p,\tilde{a},e$ is the number of lattice paths satisfying certain properties (we will denote this set by $path_e((m,n),p,\tilde{a})$). Thus, our main goal will be to find a function $\phi : \{mon((m,n),p,\tilde{a},v)\}_{v=0}^{\infty} \to$

$path_e((m,n),p,\tilde{a})_{e=0}^{\infty}$ satisfying our preimage constraint. Furthermore, given v, e we need the size of the preimage in $mon((m,n),p,\tilde{v})$ of an element in $path_e((m,n),p,\tilde{a})$ to be exactly $\binom{v-e+c}{c}$. Then by a combination of $1, 2$, and 3 we will have shown that $H_e((m,n),p,\tilde{a},v) = |mon((m,n),p,\tilde{a},v)|$.

In order to achieve this, we will define $F(v)$ and both of our pertinent sets, then construct ϕ, and finally show that our chosen ϕ satisfies our requirements.

3 Definitions

Definition: Let $X = [X_{ij}]$ be an mxn matrix of indeterminates. Let K be a commutative ring. Let $u_1, u_2, ..., u_p, r_1, r_2, ..., r_q, v_1, v_2, ..., v_p, s_1, s_2, ..., s_q$ be integers such that

$1 \leq a_1 \leq a_2 \leq ... \leq a_p \leq m \qquad 1 \leq b_1 \leq b_2 \leq ... \leq b_q \leq n$

$0 \leq r_1 \leq r_2 \leq ... \leq r_p \leq m \qquad 0 \leq s_1 \leq s_2 \leq ... \leq s_q \leq n$

Let I be the ideal generated by (r_i+1)-minors of the first a_i rows, and the (s_j+1)-minors of the first b_j columns. Any ring of the form $K[x]/I$ is called a determinantal ring.

In Kulkarni's paper we study a slight modification of determinantal rings, which I will refer to as Abhyankar Determinantal Rings (or ADR's for short).

Definition: We take the same assumptions as from determinantal rings, but we modify K and I. Firstly, instead of considering any commutative ring K, we instead limit ourselves to fields. Secondly, we let $q = p, r_i = i, s_i = i$ (note, this last definition makes sense since $q = p$ so the indices are now the same). Finally, we add in the $(p+1) \times (p+1)$-minors to generate our ideal I. Then the Abhyankar Determinantal Ring is the ring $K[X]/I$.

Essentially, our ideal I is generated by the i-minors of the first a_i rows or by b_i columns, as well as the $(p+1)$-minors of our entire matrix. Note that, in general, we will be given our a_i, and b_i in the form of a bivector $\tilde{a} = (a_1, ...a_p; b_1, ...b_p)$, and we say that $\tilde{a}$ has length p. Furthermore, the ideals I defined for ADR's are referred to as generalized determinantal ideals, and are denoted $I(p, \tilde{a})$.

Definition: Let R be a ring so that $R = \bigoplus_{i=0}^{\infty} R_i$, where each R_i is an Abelian group under our ring sum. If $R_s R_t \subseteq R_{st}$ for all $s, t \in \mathbb{N}$, then we say that R is a graded ring.

Example: Let R be a commutative ring, and let x be an indeterminate over R. Then $R[x]$ is a graded ring, by letting $R_i = \{f | f \in R, deg(f) \leq i\} \cup \{0\}$.

We will now define the Hilbert Polynomial over an ideal I. One should note however, that Hilbert Polynomials are defined more generally.

Definition: Consider the ring $R = K[x_1, ...x_n]$ where K is a field, and $x_1, ...x_n$ are indeterminates over K. Let I be some ideal of R, and let $A = R/I$. Then A is a graded ring via $A = \bigoplus_{d=0}^{\infty} A_d$ where $A_d = \{f + I | f \in R, deg(f) \leq d\}$ ($deg(0) = -1$). We note that A_d is a finite dimensional vector space over K. Let $h_I(t) = dim_K(A_d)$. This is called the Hilbert Function of I. Further, we define the power series $H_I(t) = \sum_{d=0}^{\infty} h_I(d)t^d$ to be the Hilbert Series of I.

Hilbert-Serre Theorem: Let I be an ideal of $K[x_1, ..., x_n]$. Then $H_I(t) = \frac{a_0+a_1t+...+a_kt^k}{(1-t)^{n-1}}$ with $a_i \in \mathbb{Z}$ for $i = 1, ..., k$. Moreover, $h_I(d) = \sum_{i=0}^{k} a_i \binom{x+n-i}{n} =: p_I(d)$ for sufficiently large d, and $p_I(d) \in \mathbb{Q}[x]$.

Definition: We say that p_I (as defined in the previous theorem)is the Hilbert Polynomial of I.

Definition: Let I an ideal. If $h_I(d) = p_I(d)$ for all $d \geq 0$, then we say that I is a Hilbertian Ideal.

Our next step is to define the special set of monomials referred to in the introduction.

Definition: Let $rec(m, n)$ be the set of points $[1, m] \times [1, n]$. We define a monomial on $rec(m, n)$ to be a map from $rec(m, n)$ to $\mathbb{N}$. Further, we define the index of a subset $S \subseteq rec(m, n)$ to be the maximal k such that a sequence $(a_1, b_1), (a_2, b_2), ..., (a_k, b_k)$ exists, where $1 \leq a_1 < a_2 < ... < a_k \leq m$ and $1 \leq b_1 < b_2 < ... < b_k \leq n$. Finally, we define the index of a monomial to be the index of its support, and the degree of a monomial t to be the sum over all points (x, y) in $rec(m, n)$ of $t(x, y)$. (we indicate these by $deg(S), ind(M), deg(M)$ respectively for a set S and monomial M).

Definition: Let $m, n, p \in \mathbb{Z}$, $v \in \mathbb{N}$, and $\tilde{a}$ be a bivector of length p bounded by (m, n) (bounded by (m, n) means $a_p \leq m$ and $b_p \leq n$). We define $mon((m, n), p, \tilde{a}, v)$ to be the set of monomials on $rec(m, n)$ of degree v, index less than p, and where the index of the first $a_i - 1$ rows or $b_i - 1$ columns is at most $i - 1$ (for all a_i, b_i from our bivector).

We now conclude this preliminary definition section with a description of the relevant lattice paths.

Definition: We define a lattice path on $rec(m, n)$ to be a sequence $(x_1, y_1), (x_2, y_2), .., (x_k, y_k)$ of points in $rec(m, n)$ such that $y_1 = n$, $x_k = m$, and for all $1 < i \leq n$ either
1. $x_i - x_{i-1} = -1$ and $y_i = y_{i-1}$ (a step north) OR
2. $x_i = x_{i-1}$ and $y_i - y_{i-1} = 1$. (a step west).
Furthermore, we refer to (x_1, y_1) and (x_k, y_k) as the starting and end point of our path respectively. We will frequently say that the path 'goes from (x_1, y_1)

to (x_k, y_k).

Definition: Let $\mathcal{L}$=$(x_1, y_1), ..., (x_k, y_k)$ be a lattice path on $rec(m, n)$, we say that (x_i, y_i) is a node of $\mathcal{L}$ if $(x_{i-1}, y_{i-1}), (x_i, y_i)$ is a step north, and $(x_i, y_i), (x_{i+1}, y_{i+1})$ is a step west. The singular lattice path (from a to b $(a, b \in rec(m, n))$ with no nodes is called the hook at (a, b).

Definition: We say that a p-tuple of lattice paths is non-crossing if the sequences of any two components are disjoint. Furthermore, the set of nodes of a p-tuple of lattice paths is the union of their respective nodes.

Note: Any lattice path is uniquely defined by its starting point, ending points and nodes.

Definition: Let $path_e((m, n), p, \tilde{a})$ be the set of p-tuples of non-crossing lattice paths on $rec(m, n)$ so that path $\mathcal{L}_i$ starts at a_i and ends at b_i. (As usual, we have $\tilde{a} = (a_1, ..., a_p; b_1, ..., b_p)))$.

4 Hilbert Polynomials and Monomials

This section will be a brief summary of Abhyankar's work in establishing a correspondence between the Hilbert polynomial of a generalized determinantal ideal and our special set of monomials.

Theorem: Let $R = K[X]$ be a graded ring over a field K. Let $I(p, \tilde{a})$ be a generalized determinantal ideal in R. Then:

$$dim_K(K_v/(I(p, \tilde{a})_v) = \sum_{d=0}^{\infty} (-1)^d F_d((m, n), p, \tilde{a}) \binom{v+c-d}{c-d}$$

where

$$F_d((m, n), p, \tilde{a}) = \sum_{e=0}^{\infty} \binom{e}{d} H_e((m, n), p, \tilde{a})$$

$$c = \sum_{i=1}^{p} ((m - a_i) + (n - b_i)) + p - 1$$

$$H_e((m, n), p, \tilde{a}) = \sum_{e_1+...e_p=e} \det_{1\leq j\leq n} \binom{m-a_i+i-j}{e_i+i-j} \binom{n-b_j+j-i}{e_i}$$

$e_1, ..., e_p \in \mathbb{N}$.

As one would expect, the proof of the above theorem is fairly involved - Abhyankar proves it by finding a correspondence between $dim_K(K_v/I(p, \tilde{a})_v)$ and a certain generalized class of standard young tableaux. At any rate, here is what we should take from this result. Firstly, $dim_K(K_v/I(p, \tilde{a}))$ is in fact just $h_{I(p,\tilde{a})}(v)$ by definition. Next, when fully expanded, we see that this value is simply a polynomial in v. Thus, our Hilbert function is the same as our Hilbert polynomial, and so the generalized determinantal ideals are Hilbertian. Furthermore, and most importantly, since $\sum_{d=0}^{\infty} (-1)^d \binom{e}{d} \binom{v+c-d}{c-d} = \binom{v+c-e}{c}$, we find that $F(v)$ from our 1st and 2nd facts, is actually just the Hilbert polynomial of the generalized determinantal ideal. Then, by our 1st fact we see that $F(v)$ measures $|mon((m,n),p,\tilde{a},v)|$.

5 Lattice Paths and Monomials

We've now finally layed down all the necessary definitions to understand fact 3 properly. Let's rephrase it thusly.

3'. We are given a family of sets $\{mon((m,n),p,\tilde{a},v)\}_{v=0}^{\infty}$ and a family of pairwise disjoint finite sets $\{path_e((m,n),p,\tilde{a})\}_{e=0}^{\infty}$. $\{F(v)\}_{v=0}^{\infty}$ is a family of polynomials over $\mathbb{Q}[v]$ with $F(v) = |mon((m,n),p,\tilde{a},v)|$, and $F(v) = \sum_{e=0}^{\infty} H_e((m,n),p,\tilde{a},e)\binom{v-e+c}{c}$. We note that the $\binom{v-e+c}{c}$ form a family of $\mathbb{Q}$-linearly independent polynomials (their highest degree is different for all differing choices of e), and furthermore that $\binom{v-e+c}{c}$ depends on $v,e,m,n,\tilde{a}$ but not on our specific choice of lattice path. Thus, we can now apply our plan from the introduction to show that $H_e((m,n),p,\tilde{a},v) = |mon((m,n),p,\tilde{a},v)|$.

Given a monomial M in $mon((m,n),p,\tilde{a},v)$ our function will create the sets $S_p,...S_1$ which will correspond to lattice paths $\sigma_p,...,\sigma_1$ respectively. Unfortunately the definition of S_{i-1} will depend upon σ_p. Because of this, we will do things a little backwards and show the correspondence between sets and lattice paths first, and then show how to build the actual sets afterwards.

<u>Algorithm:</u> Let $T \subseteq rec(m,n)$, and (a,b) be given. We construct $\mathcal{L}$.
0. Take (a,n) to be the starting point, and (m,b) to be the end point. Set $i_0 := a$.
1. Let $j'_k = \max\{j : (i,j) \in T, i > i_{k-1}\} \cup \{b\}$.
2. If $j'_k = b$ then take $\mathcal{L}$ the lattice path with node set $\{(i_1,j_1),...,(i_{k-1},j_{k-1})\}$. (note that if j_1 is not defined, then this set is empty, so our lattice path is simply the hook from (a,n) to (m,b)).
3. Let $j_k = j'_k$. Let $i_k = \max\{i : (i,j_k) \in T\}$.
4. If $i_k = m$, then take $\mathcal{L}$ to be the lattice path with node set $\{(i_1,j_1),...(i_k,j_k)\}$.
5. Set $k := k+1$. Go back to 1.

Let's briefly show that this algorithm does yield a lattice path, and that it will be unique. Well, by our definitions we note that the set of points $\{(i_1,j_1),...,i_k,j_k\}$ (considered as a subsequence) has some lattice path starting at (a,n) and ending at (m,b) iff $n \geq j_1 > ... > j_k > b$ and $a < i_1 < ... < i_k \leq m$. Well in Step 1 we choose j'_k so that $i > i_{k-1}$ and then in Step 3 we choose i_k to be the max i among $(i,j_k) \in S$. Thus clearly $i_k > i_{k-1}$. The proof that $j_k < j_{k-1}$ is essentially the same, and clearly the bounds involving a,b,n,m hold. Finally, we noted that every lattice path is uniquely determined by our starting and ending points along with the node set, thus our algorithm does yield a unique lattice path.

Now, we show how to build the sets $T_p, ..., T_1$.

Algorithm: Let $M \in mon((m,n), p, \tilde{a}, v)$.
0. Let $T_p = supp(M)$ (where $supp(M)$ denotes the support of M).
1. If $i = 0$ then return $T_p, ..., T_1$.
2. Let $S_i = T_i \cap \sigma_i$ (where σ_i is the lattice path corresponding to T_i, (a_i, b_i) from our previous algorithm, and considered as a set).
3. Let $T_{i-1} = T_i \setminus S_i$.
4. Set $i := i - 1$.

We thus take ϕ to be the combination of the above algorithms that takes in $M \in \{mon((m,n), p, \tilde{a}, v)\}_{v=0}^{\infty}$ and outputs $\sigma = (\sigma_1, ..., \sigma_p) \in \{path_e((m,n), p, \tilde{a})\}_{e=0}^{\infty}$.
Naturally, we'd like to check that ϕ is well-defined.
Observe that the T_i form a descending chain.
Then $S_i \cap S_j$ is empty for $i > j$ since
$S_i \cap S_j = S_i \cap (T_j \cap \sigma_{i-1}) \subseteq S_i \cap T_j \subseteq S_i \cap T_{i-1} = S_i \cap (T_i \setminus S_i) = \{\}$.
In addition, since we used T_p to output a p-tuple of paths, if we input T_k $(k < p)$ instead, we would output a k-tuple of paths by construction. Thus, we will need T_k to be the support of some monomial $N \in mon((m,n), k, (a_1, ..., a_k; b_1, ..., b_k), v)$.
Well, this will be true as long as $ind(T_k) \leq k$, else we could make more than k lattice paths from T_k.

Proposition: $ind(T_k) \leq k$.

Proof: By induction on k.
Base Case: For $k = p$, $T_p = supp(M)$ for some $M \in mon((m,n), p, \tilde{a}, v)$. Thus $ind(M) = p$. But $ind(M) = ind(T_P)$ by definition.
Now assume the result for all $k \geq i$. Assume for a contradiction that $ind(T_{i-1}) \geq i$. Well since $ind(T_i) = i$ by assumption, we have $ind(T_{i-1}) = i.$. Then we can find a sequence $(x_1, y_1), ..., (x_i, y_i)$ in T_{i-1} with $1 \leq x_1 < ... < x_i \leq m$ and $1 \leq y_1 < ... < y_i \leq n$. Then $x_i \geq a_i$ and $y_i \geq b_i$ (since by the definition of $mon((m,n), p, \tilde{a}, v)$ the index over the first $a_i - 1$ rows or $b_i - 1$ columns is at most $i-1$). Further, we have that σ_i contains a point (x, y) so that $x > x_i$ and $y > y_i$ (this is clear by our original choice of j_k, and the fact that (x_i, y_i) cannot be in path σ_i - by construction of T_i). But then, note that our sequence is also in T_i so that we can augment by (x, y). This is a contradiction since then $ind(T_i) = i+1$.

Now we draw a few conclusions to ultimately show that ϕ is surjective:
1. The p-tuple σ is non-crossing by construction. Further, any path σ_i considered as a subset of $rec(m,n)$ must have index 1. This is geometrically clear. Thus $ind(S_i) = 1$ since $S_i \subseteq \sigma_i$.
2. Because of 1, we must have $ind(T_i) = i$ since we are always removng subsets of index 1. Therefore, in particular $ind(T_0) = 0$, and so T_0 is empty.
3. The S_i form a partition of T, by construction and since T_0 is empty.

4. Due to 3, we have that $T \subseteq \sigma$ (σ is viewed by considering each σ_i as a set, and taking their union).
5. The nodes of each σ_i are contained in S_i (from the construction of σ_i), and thus the set of all nodes of σ are contained in T.

Therefore, for any lattice path $\mathcal{L} \in \{path_e((m,n),p,\tilde{a})\}_{e=0}^{\infty}$ we can take the monomial M in $mon((m,n),p,\tilde{a},v)$ where v is the number of nodes of $\mathcal{L}$, so that $supp(M)$ is the set of nodes of $\mathcal{L}$ and M maps each of these points to 1. Then by application of these algorithms to M we would obtain $\mathcal{L}$, and so ϕ is surjective.

Finally, we would like to find the exact size of $\phi^{-1}(\sigma)$ restricted to $mon((m,n),p,\tilde{a},v)$, where $\sigma \in path_e((m,n),p,\tilde{a})$. Well, the total number of points on σ is $c+1$ since path σ_i uses $(m-a_i)+(n-b_i)+1$ points. Thus, by 4, all monomials M considered must have their support a subset of these $c+1$ points. Similarly, by 5, the support must contain all e nodes. By counting all such monomials we arrive at $\binom{v-e+c}{c}$ different possiblities.

Now we are finally able to state:

Theorem: Given positive integers m,n,p, and a bivector of length p bounded by (m,n), we have
$H_e((m,n),p,\tilde{a}) = |path_e((m,n),p,\tilde{a})$.

Lastly, if we colour the nodes black or red, we find that:

Corollary: For positive integers m,n,p,d, and a bivector $\tilde{a}$ of length p bounded by (m,n), $F_d((m,n),p,\tilde{a})$ is the number of non-crossing p-tuples of lattice paths in $rec(m,n)$ based on $\tilde{a}$ with exactly d black nodes.

6 Works Consulted

"Counting of Paths and Coefficients of the Hilbert Polynomial of a Determinantal Ideal" by Kulkarni
"Enumerative Combinatorics of Young Tableaux" by Abhyankar
"A Course in Commutative Algebra" by Kemper and Gregor
"Determinantal Rings" by Bruns and Vetter